

Periodic Solutions for N -Body-Type Problems

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Abstract

We consider non-autonomous N -body-type problems with strong force type potentials at the origin and sub-quadratic growth at infinity, and using Ljusternik-Schnirelmann theory, we prove the existence of unbounded sequences of critical values for the Lagrangian action corresponding to non-collision periodic solutions.

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1 Introduction and Main Result

In the 1975 paper of Gordon ([13]), we find the first prominent use of variational methods in the study of periodic solutions of Newtonian equations with singular potential $V(t, x) \in C^1([0, T] \times (R^n \setminus S), R)$:

$$\begin{cases} \ddot{x} + V'(t, x) = 0, & x \in R^n \\ x(t+T) = x(t), \end{cases} \quad (1.1)$$

where $V(t+T, x) = V(t, x)$ satisfies the following Gordon's strong-force (SF) condition:

There exists a neighborhood N of the set S and a C^2 function U on $N - S$ such that

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- (i) $U(x) \rightarrow -\infty$ as $x \rightarrow S$,
- (ii) $-V(t, x) \geq |\nabla U(x)|^2, \forall x \in N \setminus S$.

Using variational minimizing methods, Gordon proved the following theorem:

Theorem 1.1. (Gordon) *Under the above conditions and (V_1) :*

$$V(t, x) < 0, x \neq 0,$$

we have that there exist periodic solutions which tie (wind around) S and have arbitrary given topological (homotopy) type and given period.

Ambrosetti-CotiZelati ([1],[2]) used Morse theory to generalize Gordon's result and obtained the following theorem:

Theorem 1.2. *Assume $V \in C^2([0, T] \times R^n, R)$, $V(t+T, x) = V(t, x)$ satisfies Gordon's strong force condition and (V_i) :*

$$|V(t, x)|, \quad |V_x(t, x)| \rightarrow 0$$

uniformly for all t as $\|x\| \rightarrow \infty$, and $\exists R_1 > 0$ s.t.

$$V(t, x) < 0, \quad \forall \|x\| \geq R_1,$$

then (1.1) has infinitely many T -periodic solutions.

Motivated by Gordon ([13]) and Ambrosetti-Coti Zelati ([1],[2]), Jiang M. Y. ([15]) continues the application of Morse theory to arrive at his theorem which improves the condition (V_i) :

Theorem 1.3. *Let Ω be an open subset in R^n with compact complement $C = R^n \setminus \Omega$, $n \geq 2$. Assume $V \in C^2([0, 2\pi] \times \Omega, R)$, $V(t + 2\pi, x) = V(t, x)$, and*

(A_1) . $\exists R_0$ such that

$$\sup\{|V(t, x)| + |V'_x(t, x)| \mid (t, x) \in [0, 2\pi] \times (R^n \setminus B_{R_0})\} < +\infty$$

(A_2) . V satisfies Gordon's strong force condition.

Then (1.1) has infinitely many 2π -periodic solutions.

Using Ljusternik-Schnirelman theory, Majer[16] got the following result which improves the condition (A_1) :

Theorem 1.4. *Assume $W \in C^1([0, T] \times (R^N \setminus \{0\}), R)$ satisfies*

$$(i). \quad W(t + T, x) = W(t, x).$$

$$(ii). \quad \exists c \in R, \theta < 2, r > 0, \text{ such that}$$

$$W(t, x) \leq c|x|^\theta, \quad W'(t, x)x - 2W(t, x) \leq c|x|^\theta, \quad \forall |x| > r, \forall t > 0.$$

$$(iii). \quad \text{If } a < \left(\frac{\pi}{T}\right)^2$$

Then

$$\ddot{u} + au + W'(t, u) = 0$$

has infinitely many T -periodic solutions.

For 3-body type problem, Bahri-Rabinowitz ([3]) used Morse theory to prove:

Theorem 1.5. (Bahri-Rabinowitz) *Let $V(q) = \frac{1}{2} \sum_{1 \leq i \neq j \leq 3} V_{ij}(q_i - q_j)$. Assume V_{ij} satisfies*

$$(V_1). \quad V_{ij} \in C^2(R^l \setminus \{0\}, R).$$

$$(V_2). \quad V_{ij} < 0.$$

$$(V_3). \quad V_{ij}(q), V'_{ij}(q) \rightarrow 0 \text{ as } |q| \rightarrow \infty.$$

$$(V_4). \quad V_{ij}(q) \rightarrow -\infty \text{ as } q \rightarrow 0.$$

$$(V_5). \quad \text{For } \forall M > 0, \exists R > 0, \text{ s.t.}$$

$$V'_{ij}(q) \cdot q > M|V_{ij}(q)|, \quad |q| > R.$$

$$(V_6). \quad \exists U_{ij} \in C^1(R^l \setminus \{0\}, R), \text{ s.t.}$$

$$U_{ij}(q) \rightarrow \infty \text{ as } q \rightarrow 0, \text{ and } -V_{ij} \geq |U'_{ij}|^2.$$

Then for any given $T > 0$,

$$\ddot{q}_i + \frac{\partial V(q)}{\partial q_i} = 0 \tag{1.2}$$

has infinitely many T -periodic noncollision solutions.

We say that a function $X(t) = (x_1(t), \dots, x_N(t)) \in C^2(R, (R^k)^N)$ is a non-collision T -periodic solution of (1.2) if $X(t)$ satisfies $x_i(t) \neq x_j(t)$ for all $i \neq j$ and $t \in R$, and satisfies the equation (1.2) and is indeed T periodic.

Majer-Terracini ([17]) generalized the result of Bahri-Rabinowitz to n -body type problems:

$$\ddot{x}_i(t) + \nabla_{x_i} V(t, x_1(t), \dots, x_N(t)) = 0, \quad x_i(t) \in R^k, \quad i = 1, \dots, N. \quad (1.3)$$

They proved the following theorem:

Theorem 1.6. *Assume $k \geq 3$, and $V_{ij} \in C^1((R^k \setminus 0) \times R, R)$ are T -periodic in t , V satisfies*

$$(V_1). \quad V_{ij}(t, x) = V_{ji}(t, -x), \forall x \in R^k \setminus \{0\}.$$

$$(V_2). \quad V_{ij}(t, x) \leq 0, \forall x \in R^k \setminus \{0\}.$$

$$(V_3). \quad V_{ij}(t, \xi) \rightarrow -\infty \text{ uniformly in } t \text{ as } |\xi| \rightarrow 0, \text{ for all } 1 \leq i \neq j \leq N, \text{ and } V_{ij} \text{ satisfies Gordon's strong force condition.}$$

$$(V_4). \quad \exists \rho > 0, \exists \theta \in [0, \frac{\pi}{2}) \text{ s.t. any } (\nabla V_{ij}(t, x), x) \leq \theta, \forall x, |x| > \rho.$$

Then (1.3) has at least one T -periodic non-collision solution.

For symmetrical potentials, Fadell-Husseini[11], Zhang-Zhou[25] proved that

Theorem 1.7. *We assume V_{ij} satisfies the following conditions:*

$$(V1). \quad V(t, x) = \frac{1}{2} \sum_{1 \leq i \neq j \leq N} V_{ij}(t, x_i - x_j).$$

$$(V2). \quad V_{ij} \in C^1(R \times (R^k - \{0\}); R), \text{ for all } 1 \leq i \neq j \leq N.$$

$$(V3). \quad V_{ij}(t, \xi) \rightarrow -\infty \text{ uniformly in } t \text{ as } |\xi| \rightarrow 0, \text{ for all } 1 \leq i \neq j \leq N.$$

$$(V4). \quad V_{ij}(t, \xi) \leq 0, 1 \leq i \neq j \leq N, \xi \neq 0.$$

$$(V5). \quad \text{the strong force condition (see [13]) holds for } V_{ij}.$$

$$(V6). \quad V_{ij}(t + T/2, -\xi) = V_{ij}(t, \xi).$$

Then there exist unbounded sequences of critical values for the Lagrangian action corresponding to non-collision periodic solutions for (1.3).

In this paper, we consider a relaxation of condition (V4) which required the potentials to be non-positive, but still maintain the potentials have some growth so that the result in Theorem 1.7 still holds. We use Majer's abstract critical point theorem to study the N-body-type problem. The key difficulty is in proving the local Palais-Smale condition, but we are able to secure the following:

Theorem 1.8. *Assume V_{ij} satisfies (V1) – (V3), (V5), (V6) and*
 $(V4)' \exists g > 0, \theta < 2, r > 0, \text{ s.t.}$

$$V_{ij}(t, \xi) \leq gm_i m_j |\xi|^\theta, |\xi| > r.$$

Then there exist unbounded sequences of critical values for the Lagrangian action corresponding to non-collision periodic solutions for (1.3).

Notice that the condition $(V4)'$ in our theorem 1.8 is a kind of growth condition which weak the ordinary condition on potentials which need non-positive, we have the following Corollary:

Corollary 1.9. *Let $\alpha \geq 2; r_1 > 0, r_2 > r_1; a, g > 0, \theta < 2$ and*

$$V(t, x) = \frac{1}{2} \sum_{1 \leq i \neq j \leq N} V_{ij}(x_i - x_j).$$

$$V_{ij}(\xi) \in C^1(R^k - \{0\}, R)$$

satisfies that

$$V_{ij}(\xi) = -am_i m_j |\xi|^{-\alpha}, |\xi| < r_1;$$

$$V_{ij}(\xi) = gm_i m_j |\xi|^\theta, |\xi| \geq r_2 > r_1.$$

Then the assumptions and the result of Theorem 1.8 holds.

We also notice that for Newtonian type potentials, there were a deep and lively literature; for example, [4], [6]-[10], [12]-[13], [18]-[24].

2 Some Lemmas

We introduce spaces

$$E = \{(x_1, \dots, x_N) | x_i \in H^1(R/TZ; R^k), x_i(t + T/2) = -x_i(t)\},$$

$$\Delta = \{(x_1, \dots, x_N) | x_i \in H^1(R/TZ; R^k), x_i(t) \neq x_j(t), \forall t, i \neq j\},$$

where $H^1(R/TZ; R^k)$ is the metric completion of smooth T -periodic functions for the norm $\|x\|_{H^1} = \left(\int_0^T |x(t)|^2 + |\dot{x}(t)|^2 dt \right)^{1/2}$, and the functional $f : \Delta \rightarrow R$ is defined by

$$f(x_1, \dots, x_N) = \sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i(t)|^2 dt - \int_0^T V(t, x_1(t), \dots, x_N(t)) dt.$$

Clearly, E is a closed subspace of $H^1(R/TZ; (R^k)^N)$, and so a Hilbert space, while Δ is an open subset of E .

Using a standard argument (for instance, see [25]), it is easy to prove the following Lemma 2.1:

Lemma 2.1. *Suppose (V1) – (V2) and (V6) hold, then a critical point of f in Δ is a non-collision solution of (1.3).*

The closed subset $\Gamma = E - \Delta$ of E will be called the collision set, and a standard argument can be applied to show that the strong force assumption (V3) implies that $f(X) \rightarrow +\infty$ when X approaches the collision set Γ . More precisely, we have the following lemma.

Lemma 2.2. *([13]/[25]) Assume V satisfies (V1)-(V3) and (V5). Let $\{X^n\}$ be a sequence in Δ and $X^n \rightarrow X \in \Gamma$ in both the C^0 topology and weak topology of E , then $f(X^n) \rightarrow +\infty$.*

Lemma 2.3. *Assume V satisfies (V1) – (V3), $(V4)'$, (V5) and (V6), then there is a constant λ_0 depending on g, m_i, r, θ , s.t. f satisfies the $(PS)_c$ condition for $c \geq \lambda_0$; that is, any sequence $\{x^k\} \subset \Delta$ satisfying $f(x^k) \rightarrow c$ and $f'(x^k) \rightarrow 0$ is pre-compact in H^1 .*

Proof. We notice that the arguments in Jiang [15] (also Chang K.C.[5]) and Majer [16] cannot be directly generalized to the N-body case because of the translation invariance for positions in N-body problems. Here we must consider the differences and must use different arguments. By Holder's inequality, we have

$$\sum_{i < j} m_i m_j |x_i - x_j|^\theta \leq \left(\sum_{i < j} m_i m_j \right)^{(2-\theta)/2} \left(\sum_{i < j} m_i m_j |x_i - x_j|^2 \right)^{\frac{\theta}{2}}$$

Calculating,

$$\begin{aligned}
\sum_{i < j} m_i m_j |x_i - x_j|^2 &= \frac{1}{2} \sum_{1 \leq i, j \leq N} m_i m_j |x_i - x_j|^2 \\
&= \sum_{i=1}^N m_i \sum_{i=1}^N m_i |x_i|^2 - \left(\sum_{i=1}^N m_i x_i \right)^2 \\
&\leq \sum_{i=1}^N m_i \sum_{i=1}^N m_i |x_i|^2
\end{aligned}$$

and

$$f(x^k) = f(x_1^k, \dots, x_N^k) = \sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt - \int_0^T V(t, x_1^k(t), \dots, x_N^k(t)) dt.$$

Let

$$\xi_{ij}^k(t) = x_i^k(t) - x_j^k(t)$$

We consider the three possibilities:

(i). For all $1 \leq i, j \leq N$ and for all $t \in [0, T]$, $|\xi_{ij}^k(t)| > r$ when k is large, then by (V4)' and the above inequality, we have

$$\begin{aligned}
&\sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt - \int_0^T V(t, x_1^k(t), \dots, x_N^k(t)) dt \\
&\geq \sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt - g \left(\sum_{i < j} m_i m_j \right)^{(2-\theta)/2} \left[\sum_{i=1}^N m_i \right]^{\frac{\theta}{2}} \int_0^T \left[\sum_{i=1}^N m_i |x_i^k|^2 \right]^{\frac{\theta}{2}} dt.
\end{aligned}$$

Since $x^k(t + T/2) = -x^k(t)$ implies $\int_0^T x^k(t) dt = 0$, by the Wirtinger's inequality and $f(x^k) \rightarrow c \leq d$, we get

$$d \geq \sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt - g \left[\sum_{i < j} m_i m_j \right]^{(2-\theta)/2} \left[\sum_{i=1}^N m_i \right]^{\frac{\theta}{2}} \left[\frac{T}{2\pi} \right]^\theta \int_0^T \left[\sum_{i=1}^N m_i |\dot{x}_i^k|^2 \right]^{\frac{\theta}{2}} dt.$$

By the assumption (V4)', we know that $\theta < 2$, hence we have $e > 0$ such that

$$\sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt \leq e.$$

(ii). There are $1 \leq i_0, j_0 \leq N$ such that for all $t \in [0, T]$, there holds $|\xi_{i_0 j_0}^k(t)| \leq r$ when k is large, then by Lemma 2.2 and (V2), we have $a > -\infty$ and $0 < b < +\infty$ such that for all $t \in [0, T]$,

$$a \leq V_{i_0 j_0}(t, \xi_{i_0 j_0}^k(t)) \leq c m_{i_0} m_{j_0} |\xi_{i_0 j_0}^k(t)|^\theta \leq b$$

Then for the rest index pairs (i, j) and the corresponding potentials, we can use the above arguments of (i) and notice that we can add some negative terms to estimate the lower bound for the sum of all the potentials satisfying $|\xi_{ij}^k(t)| > r$:

$$-g\left(\sum_{i<j} m_i m_j\right)^{(2-\theta)/2} \left[\sum_{i=1}^N m_i\right]^{\frac{\theta}{2}} \left[\sum_{i=1}^N m_i |x_i^k|^2\right]^{\frac{\theta}{2}}.$$

Now we can consider all cases for the index pairs. We have

$$V(t, x_1^k(t), \dots, x_N^k(t)) \geq \frac{N^2 - N}{2}(-b) - g\left(\sum_{i<j} m_i m_j\right)^{(2-\theta)/2} \left[\sum_{i=1}^N m_i\right]^{\frac{\theta}{2}} \left[\sum_{i=1}^N m_i |x_i^k|^2\right]^{\frac{\theta}{2}}.$$

Then taking the integral and using a similar argument as in (i), we can also get $e_1 > 0$ such that

$$\sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt \leq e_1.$$

(iii). There are $1 \leq i_0, j_0 \leq N$, $t_1 \in [0, T]$ and $t_2 \in [0, T]$ such that $|\xi_{i_0 j_0}^k(t_1)| > r$, $|\xi_{i_0 j_0}^k(t_2)| \leq r$ when k is large.

Then

$$\begin{aligned} -V(t_1, x_1^k(t_1), \dots, x_N^k(t_1)) &\geq -g\left(\sum_{i<j} m_i m_j\right)^{(2-\theta)/2} \left[\sum_{i=1}^N m_i\right]^{\frac{\theta}{2}} \left[\sum_{i=1}^N m_i |x_i^k(t_1)|^2\right]^{\frac{\theta}{2}}, \\ -V(t_2, x_1^k(t_2), \dots, x_N^k(t_2)) &\geq -b. \end{aligned}$$

Hence for all $t \in [0, T]$, we have

$$-V(t, x_1^k(t), \dots, x_N^k(t)) \geq -b - g\left(\sum_{i<j} m_i m_j\right)^{(2-\theta)/2} \left[\sum_{i=1}^N m_i\right]^{\frac{\theta}{2}} \left[\sum_{i=1}^N m_i |x_i^k(t)|^2\right]^{\frac{\theta}{2}}.$$

Again, after taking the integral, we can find $e_2 > 0$ such that

$$\sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt \leq e_2.$$

In all cases, we get the bounded property for

$$\sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i^k(t)|^2 dt.$$

This implies $\{x^k\}$ has a weakly convergent subsequence. To prove the strongly convergent property is more or less standard. \square

The following is an abstract critical point theorem which we will use in the proof of our main result. A proof of this theorem can be found in Majer[16].

Lemma 2.4. *Let Δ be an open subset in a Banach space and let $Cat(\Delta)$ denote the category of Δ . Suppose f is a functional on Δ . Assume that*

1. $Cat\Delta = +\infty$,
2. *For any sequence $\{q_n\} \subset \Delta$ and $q_n \rightarrow q \in \partial\Delta$, we will have $f(q_n) \rightarrow +\infty$,*
3. *For any $K \in R$, $Cat_\Delta(\{q \in \Delta | f(q) \leq K\}) < +\infty$, and*
4. *There exists a $\lambda_0 \in R$ such that the Palais-Smale condition holds on the set $\{q \in \Delta | f(q) \geq \lambda_0\}$.*

Then f possesses an unbounded sequence of critical values.

Fadell-Husseini [11] and Zhang-Zhou [25] proved:

Lemma 2.5. *If Δ refers to the open subset defined in our proof of Theorem 1.8, then*

$$Cat(\Delta) = +\infty.$$

We notice that we can use the similar methods as in Lemma 2.3 to prove

Lemma 2.6. *For any $K \in R$ such that $f(q) \leq K$, there is $A \geq 0$ such that*

$$\sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i(t)|^2 dt \leq A.$$

Zhang-Zhou [25] gave:

Lemma 2.7. *For any constant $K \geq 0$, the set $D_K = \{X \in \Delta | \|\dot{X}\|_{L^2} \leq K\}$ is of finite category in Δ , i.e., $Cat_\Delta(D_K) < +\infty$.*

By the monotone property of category and using Lemma 2.6 and Lemma 2.7, we have

Lemma 2.8. *For any $K \in R$, $Cat_\Delta(\{q \in \Delta | f(q) \leq K\}) < +\infty$.*

The proof of Theorem 1.8 now follows by Lemmas 2.1-2.5 and Lemma 2.8.

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